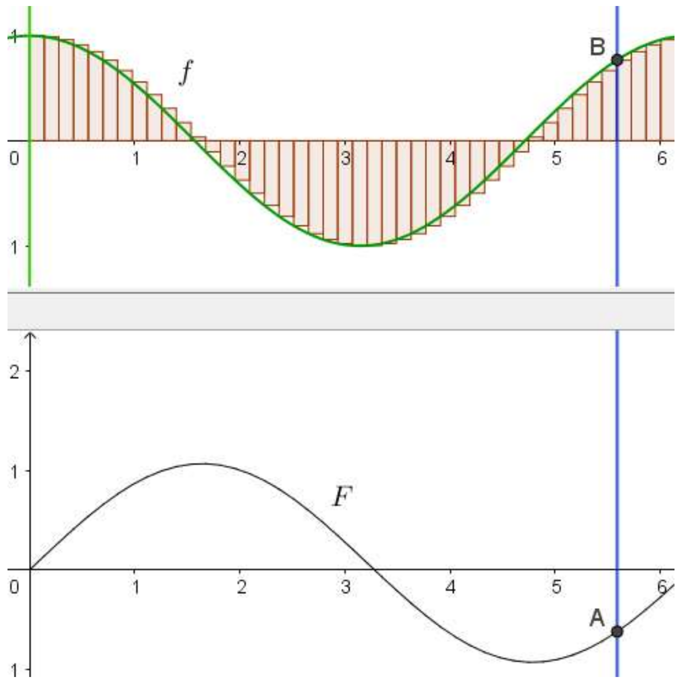
Trajectory Outline

In the introduction our students were guided to develop a way of thinking about accumulation based on a dynamic, rather than static, understanding of functions. From here, the development could progress in different directions, but given that the take home question pointed to the FTC, we can take a non-standard approach through that route. At this entry point, though, the emphasis is more on intuition than rigorous proof.

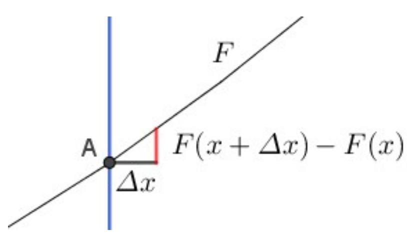
In our wrap-up we asked the students to find the derivative of the accumulation function. That is, at what rate does the accumulation function change? Coming at it this way makes perfect sense if the class just wrapped up the world of derivatives. We might consider point A in the diagram below:



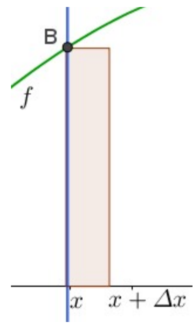
Let's also name the accumulation function "*F*" and the velocity function "f". We need to ponder:

$$\frac{F(x+\Delta x)-F(x)}{\Delta x}$$

This is nothing new for us. We can quickly envision zooming in on point A to see $F(x+\Delta x)-F(x)$ and Δx:



However, there's an extra relationship to explore when we connect our thinking to the velocity function f. $F(x)$ represents the accumulation of all the rectangles we constructed under f up to our highlighted x value. $F(x + \Delta x)$ represents all of the accumulations up to $x + \Delta x$. As such $F(x + \Delta x) - F(x)$ represents only the area of the next rectangle:



That is, the next rectangle has a base of Δx and a height of $f(x)$, so the area of that rectangle can also be represented by $f(x)\Delta x$. This means that:

$$F(x + \Delta x) - F(x) = f(x)\Delta x$$

Or

$$\frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

The rate of change over a step on our accumulation function is the value of the function! Now, here's what's amazing. Building the accumulation function takes a lot of work. So much so that we offloaded the hundreds of calculations to an applet. What happens, though, as $\Delta x \to 0$? Well, for one the accumulation curve becomes more and more accurate. That is, it represents the ideal curve. The ideal curve is one where:

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x)$$

Or, put another way:

$$F'(x) = f(x)$$

Did you catch that? The derivative of the ideal accumulation function is the original function! If I want the ideal accumulation function, I just have to find a function whose derivative is the function I was given. I don't need to do hundreds of calculations! We became good at finding derivatives, now we have a need to undo them. We need…anti-derivatives. (We will make this all more formal as time goes on.) Our work here will be used to support formal development of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

We have set a foundation and a motivation to explore accumulation functions. Accumulations, after all, can be easy to find if we can come up with the function that when I take the derivative, I end up with the function at hand. The approach from here can now easily join a standard approach to defining the antiderivative symbolically through Riemann sums.

Ted Coe, March 2020.