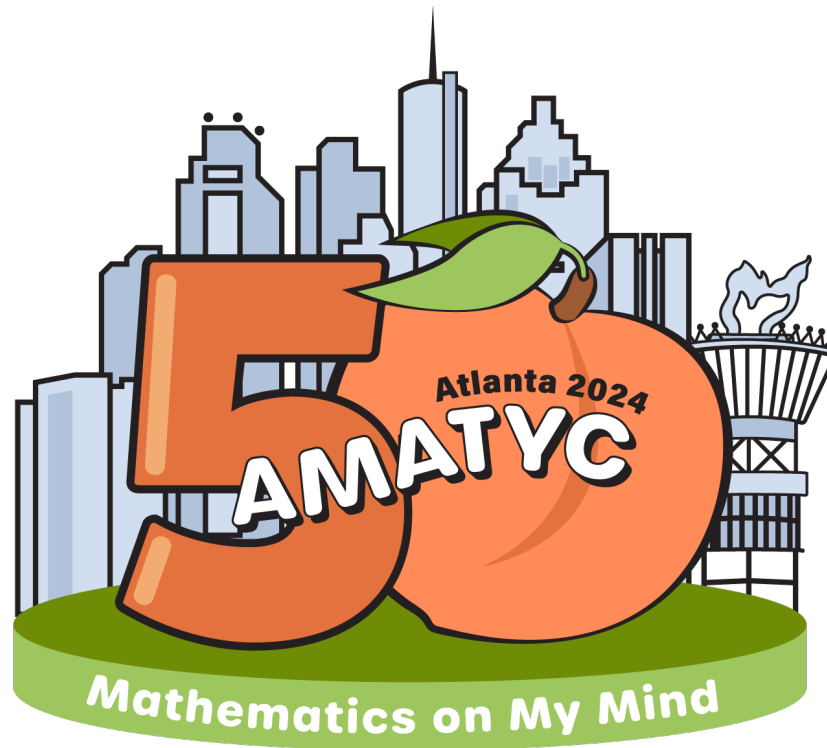


Meaningful Mathematics & Mathematical Meanings: What do *you* mean?

Ted Coe, Ph.D.

coequal
mathematics

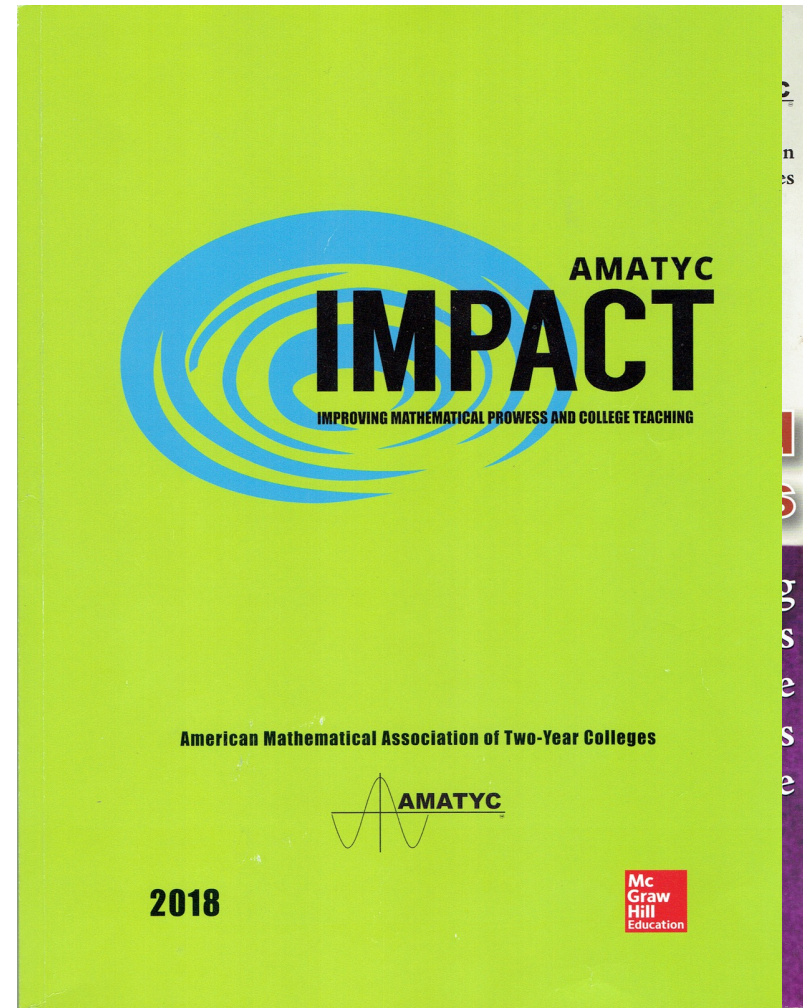


1974

2000

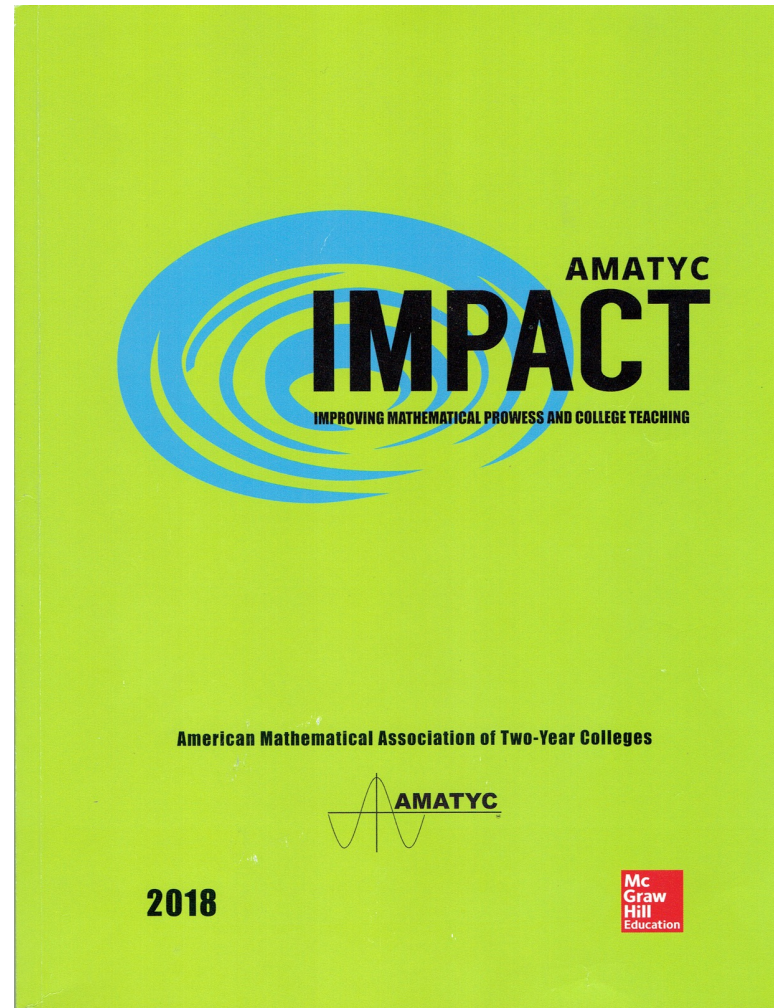
2001

2002



PROWESS

- PRoficiency
- OWnership
- Engagement
- Student Success



Meaningful mathematics and
mathematical meanings:

What do *you* mean?

“Sweet dreams are made of cheese.”

-not the Eurythmics

What does mean mean?

- How many different meanings can you imagine?
 - Noun! Verb! Adjective!
 - Dictionary.com: 26 different things.



mean¹



verb (used with object) (7)

verb (used without object) (1)

mean²



adjective (11)

mean³



noun (6)

adjective (1)

Meaningful mathematics and
mathematical meanings:

What do *you* mean?

“The ants are my friends, they’re blowin’ in the wind .”
-*not* Bob Dylan

What is rigor?

Definition

- 1 a (1) : **harsh inflexibility** in opinion, temper, or judgment : **SEVERITY** (2) : the quality of being **unyielding** or inflexible : **STRICTNESS** (3) : severity of life : **AUSTERITY**
b : an act or instance of strictness, severity, or **cruelty**
- 2 : **a tremor caused by a chill** 
- 3 : a condition that **makes life difficult**, challenging, or uncomfortable; *especially* : extremity of cold
- 4 : strict precision : **EXACTNESS** • logical *rigor*
- 5 a *obsolete* : **RIGIDITY, STIFFNESS**
b : rigidness or torpor of organs or tissue that prevents response to stimuli
c : **RIGOR MORTIS** 

Mathematical Rigor Mortis

- “Rigor” that kills all mathematical interest.
- “Rigor” that kills other mathematical options. (See also: Graduation, admission, and placement requirements)

Working together, we can find a cure.

Dana Center on Rigor

- The use of logical deductions to prove theorems
- Adhering to traditionally prescribed content
- Increased difficulty, challenging content
- Rigor and college algebra may be used interchangeably.



What Is Rigor in Mathematics Really?

Whenever major developments occur in any discipline, unintended consequences arise that need to be addressed. When the focus for entry-level mathematics shifted from access to success over a decade ago, it catalyzed the advocacy and implementation of accelerated multiple mathematics pathways alongside the algebraic-intensive pathway. Subsequent increases in success have been exciting, showing upwards of three times the success rates for students in one third of the time for some programs. With these startling increases came a widespread concern about maintaining rigor within the discipline.

In response to this concern, the Charles A. Dana Center engaged in a study of the meaning and intention of rigor in mathematics education. This paper first explores the meaning of rigor in mathematics education through the synthesis of interviews with leading mathematicians and educators, and presents a review of the literature in higher education and K-12. It concludes by offering recommendations for a shared definition of rigor and its implications for curriculum and instruction.

THE INCREASING IMPORTANCE OF MATHEMATICS

In the past 20 years, mathematics has become increasingly important to a growing number of fields of study and their related professions. In 1998, the National Science Foundation released its *Report of the Senior Assessment Panel for the International Assessment of U.S. Mathematical Sciences*, which listed 11 fields of study that interfaced with mathematics including physics, chemistry, economics, and manufacturing. The National Research Council's 2013 report, *The Mathematical Sciences in 2025*, expanded this number to 21 fields and predicted continued

Dana Center
Mathematics
PATHWAYS

Dana Center on Rigor

- “...we conclude that rigor in mathematics is a set of skills that centers on the communication and use of mathematical language.”



What Is Rigor in Mathematics Really?

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Dana Center
Mathematics
PATHWAYS

The Shifts of the Common Core

-  **Focus** strongly where the standards focus.
-  **Coherence:** Think across grades and link to major topics within grades.
-  **Rigor:** In major topics, pursue conceptual understanding, procedural skill and fluency, and application with equal intensity.

“For new math pathways to take hold, outdated notions about rigor must be reconsidered, so that high schools and colleges can ensure that students can acquire the quantitative skills they most need.”

What does a mathematically rigorous experience require?

Ohio's High School Math Pathways

Rigor: Students use mathematical language to communicate effectively and describe their work with clarity and precision. Students demonstrate how, when and why their procedures work and why they are appropriate. Students can answer the question, "How do we know?"

Rigorous courses are...	vs.	Rigorous courses are not...
Defined by complexity, which is a measure of the thinking, action or knowledge that is needed to complete the task.		Characterized by difficulty, which is a measure of effort required to complete a task.
Measured in depth of understanding.		Measured by the amount of work.
Opportunities for precision in reasoning, language, definitions and notation that are sufficient to appropriate age/course.		Based on procedure alone.
Determined by students' processes.		Measured by assigning difficult problems.



<https://bit.ly/omathrigor>

“There’s nothing that a hundred men on Mars could ever do.”

-not Toto

What does it mean to say a student is *mathematically proficient*?

What does it mean to learn mathematics?

Allow Me to Explain...

Mathematics learning, at a minimum, means learning:

- Ways of thinking that are empowered by...
- Habits of thinking, along with an ability to make sense of and use some...
- Ways of doing to do something bigger with mathematics.

Ways of Thinking (Example)

- How many years was it from December 7, 1941, to December 7, 2014?
- How did you figure it out?

Step 1: Notice the subtraction exercise.

$$2014 - 1941$$

Step 2: Set up the problem in a template.

$$\begin{array}{r} 2014 \\ -1941 \\ \hline \end{array}$$

Step 3: Work from right to left. Subtract 1 from 4.

$$\begin{array}{r} 2014 \\ -1941 \\ \hline 3 \end{array}$$

Step 4. Got it. Now move one column to the left. Subtract 4 from 1.

Step 5. Realize that I cannot subtract 4 from 1.

Step 6. Remember that I need to “borrow” from the next column to the left.

Step 7. Notice there is nothing to borrow. (Perhaps also consider how I wasn’t going to ever give it back anyway.)

Step 8. Borrow from the 2 in the far left column. Make it a 1.

Step 9. Think of the “0” in the top row as a 10. Wedge a tiny 1 between the 2 and the 0. Some might want to make it a little higher than the 0.

Step 10. Now borrow from the newly made 10. Make it a 9 by crossing out the 0 and 1 you just created during the previous step.

Step 11. Wedge a tiny 1 to the left of the original 1. I now have something that looks like this:

$$\begin{array}{r} 19 \\ \cancel{20}14 \\ - 1941 \\ \hline 3 \end{array}$$

Step 12. Subtract 4 from the newly-made 11. Get 7.

Step 13. Subtract 9 from 9.

Step 14. Subtract 1 from 1.

Step 15. Declare the answer to be 73.

What does it mean to learn mathematics?



“All the lonely Starbucks lovers, they'll tell you I'm insane”
-*not* Taylor Swift

College Ready?

What did it mean to be “college ready” in mathematics in 2010?

What will it mean to be “college ready” in mathematics in 2030?

CBMS 1970

Table 4.4

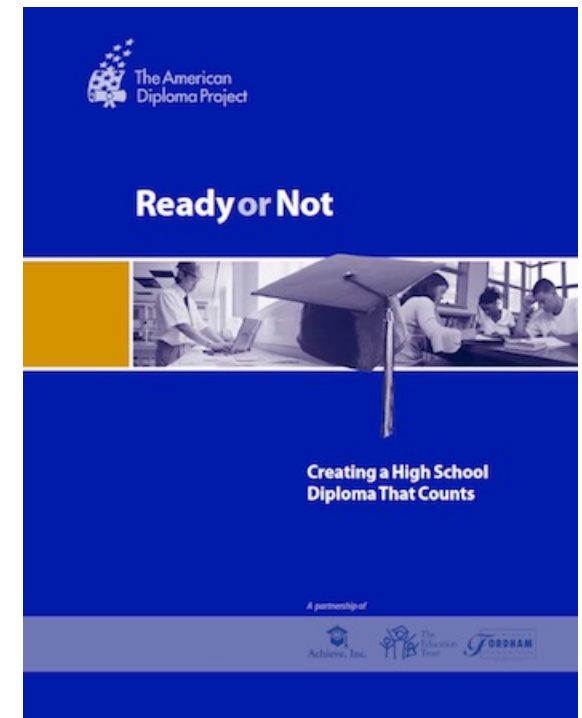
PERCENTAGES OF INSTITUTIONS HAVING SOME MATHEMATICAL
SCIENCE COURSE AS AN INSTITUTION-WIDE
GRADUATION REQUIREMENT

	1960-61	1965-66	1970-71
Universities			
Public Colleges			
Private Colleges			

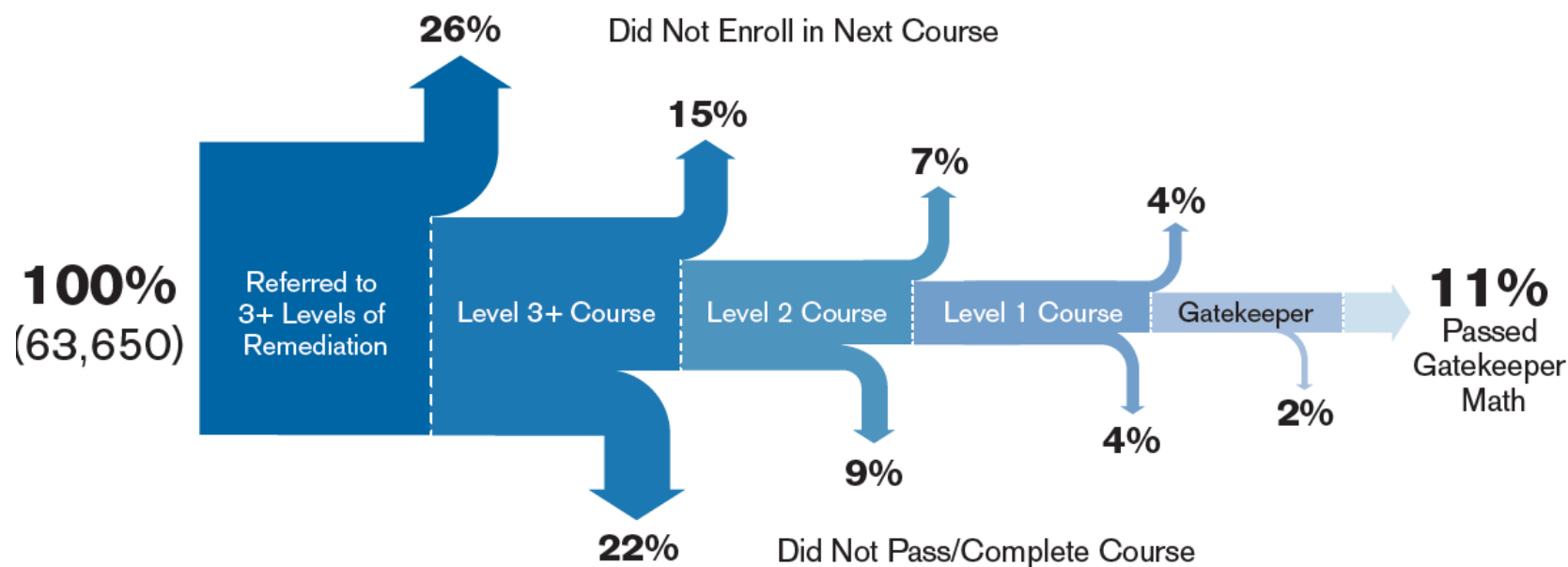
An institution-wide mathematical science requirement was favored by the mathematics chairmen in only 10 percent of universities, 15 percent of public colleges and 4 percent of private colleges.

American Diploma Project (ADP, 2004)

- The project found an unprecedented convergence between the knowledge and skills **employers** seek in new workers and those that **college faculty** expect of entering students. Both groups expect that high school graduates can... apply the higher-level math concepts historically taught in **Algebra II**, for example.

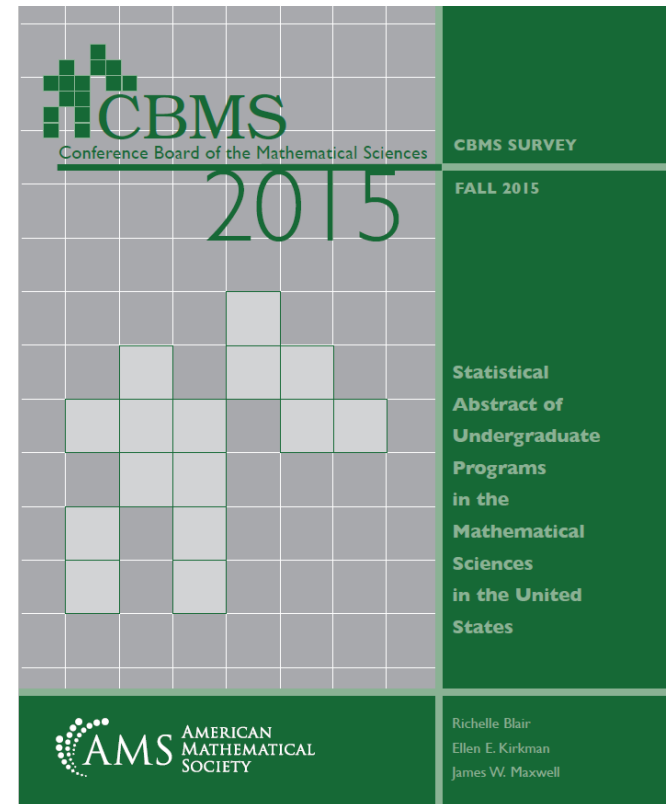


Teachers College, Columbia University (2014)

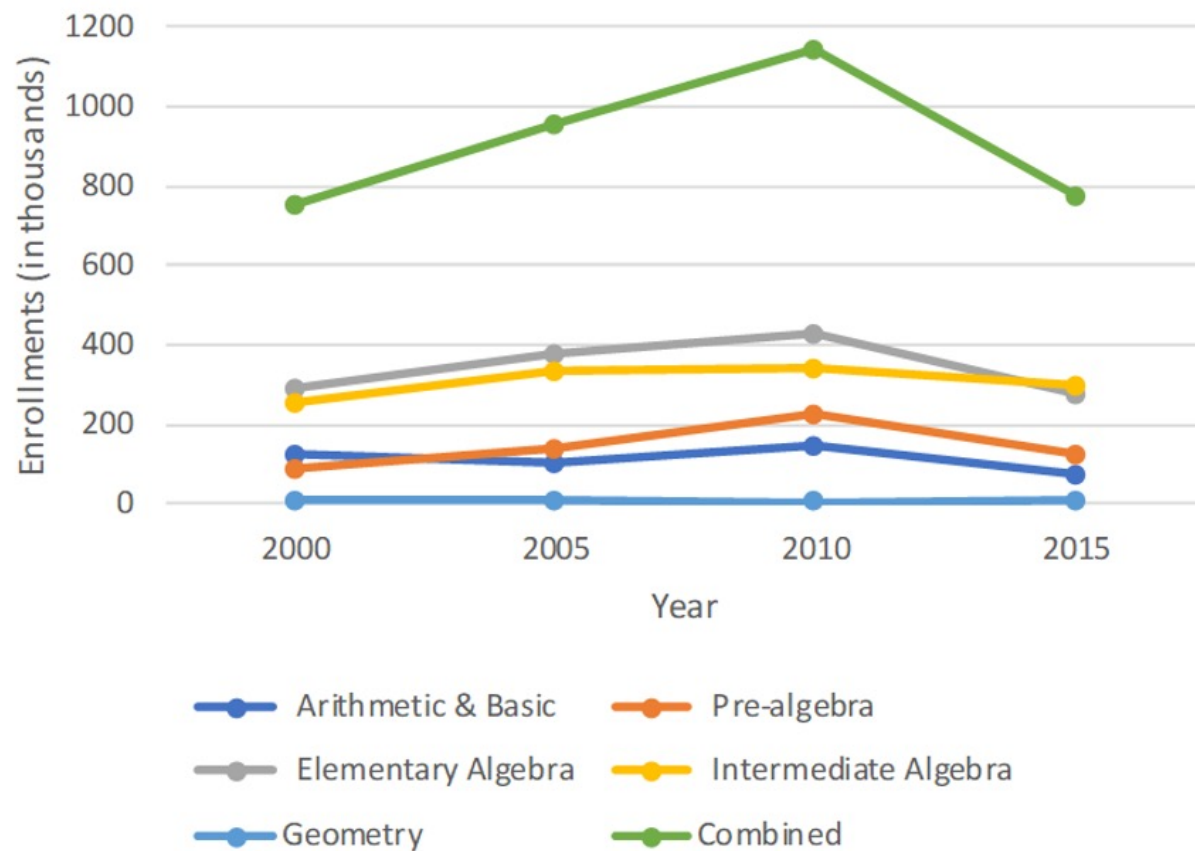


CBMS 2015 Survey

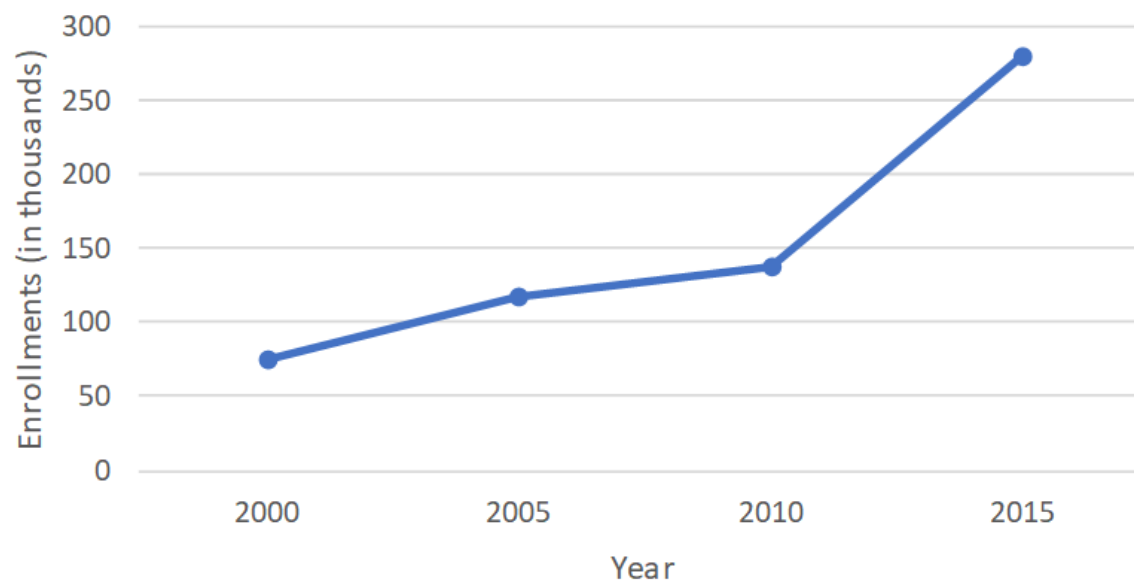
- In fall 2015, 58% of two-year colleges reported having implemented a pathways sequence



Two-year pre-college level enrollments



Two-year college
Statistics and Probability
enrollments.

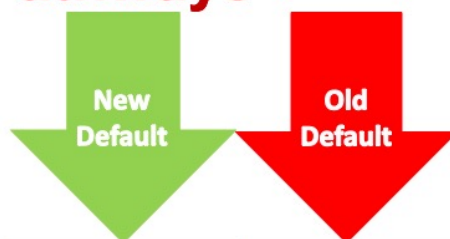


AMATYC (2014)

- The equivalent content in intermediate algebra courses is not required to master the content for most college-level mathematics courses that do not lead to calculus;

University System of Georgia

Math Pathways



Who?	All majors other than those listed to the right	Majors that require calculus at some point in the sequence	Science, Technology, Mathematics majors	Engineering majors and all Georgia Tech students
Area A2 Math course>>	Quantitative Reasoning OR Mathematical Modeling	College Algebra	Pre-calculus OR Trigonometry	Calculus
Next Math course>>	Statistics	Pre-calculus>>> Calculus	Calculus	More Calculus

University System of Georgia

Start Smart – Choose the Right MATH Pathway!

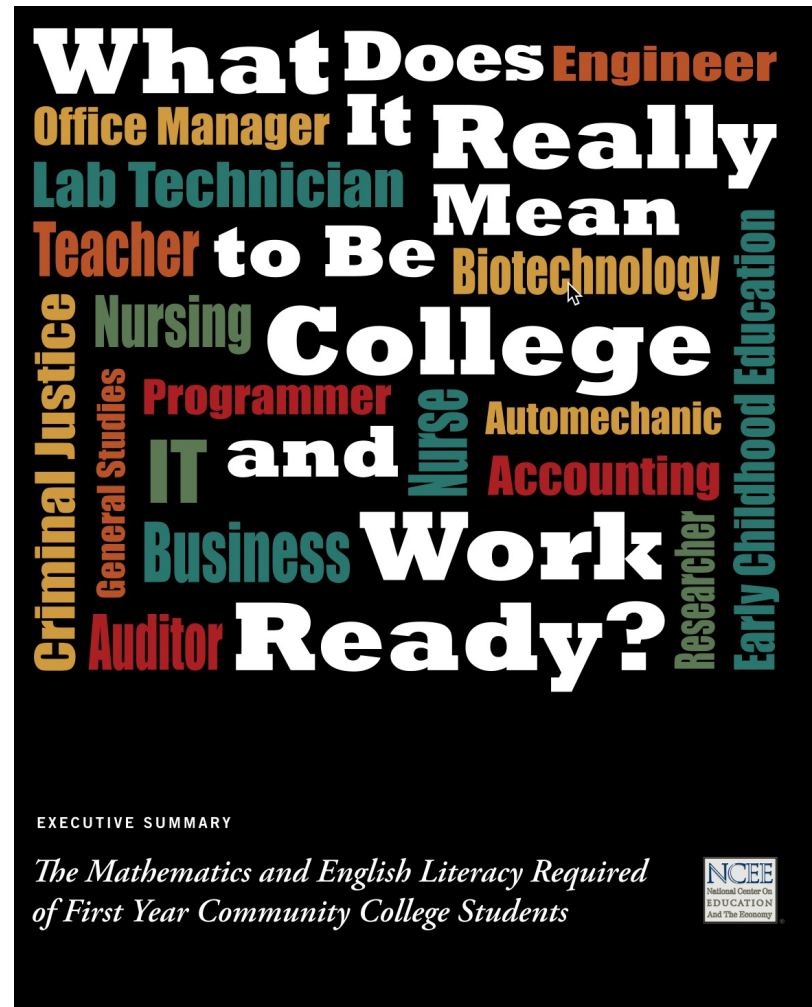
Math Course	 MATH 1001 Quantitative Reasoning OR MATH 1101 Introduction to Mathematical Modeling (OR MATH 1401 Elementary Statistics¹)		 MATH 1111 College Algebra	 MATH 1112 College Trigonometry OR MATH 1113 Pre-calculus	 Calculus
What majors start here?	Anthropology Business Administration ² Communication Criminal Justice ¹ Dental Hygiene Digital Media Educator Preparation Birth-to-Five Early Childhood Middle Grades (except Math and Science) P-12 Programs (except Health & Physical Education) Secondary (except Math and Science) English Film/Media Studies Foreign Languages General Studies ¹ Health Information Management ³	History ¹ International Studies ¹ Journalism/ Mass Communication ¹ Music Nursing Occupational Therapy Philosophy ¹ Political Science ¹ Psychology ¹ Public Relations ¹ Recreation Respiratory Therapy Social Work ¹ Sociology ¹ Speech Theatre Visual Arts	Business Administration ² Exercise and Health Science Exercise Science Geography (B.A.) Health and Physical Education (P-12) Health Information Management ³ Information Science Information Technology Logistics ⁴ Radiological Sciences	Agricultural Science ⁵ Architecture Astronomy Biology Chemistry Computer Science Educator Preparation Mathematics (Middle Grades or Secondary) Science (Middle Grades or Secondary) Engineering Technology Environmental Science Forestry Geography (B.S.) Geology Mathematics Pharmacy Physical Therapy Physics	Engineering ALL Georgia Tech students
¹ Students in these majors may take MATH 1001, MATH 1101, or MATH 1401. ² Business Administration recommendations are institution-specific. See the chart below for Business Administration recommendations for each institution. ³ Some Health Information Management programs require MATH 1111 College Algebra. Students should check institutional catalogs. ⁴ Some Logistics programs require MATH 1111 College Algebra. Students should check institutional catalogs.					

ACCUPLACER

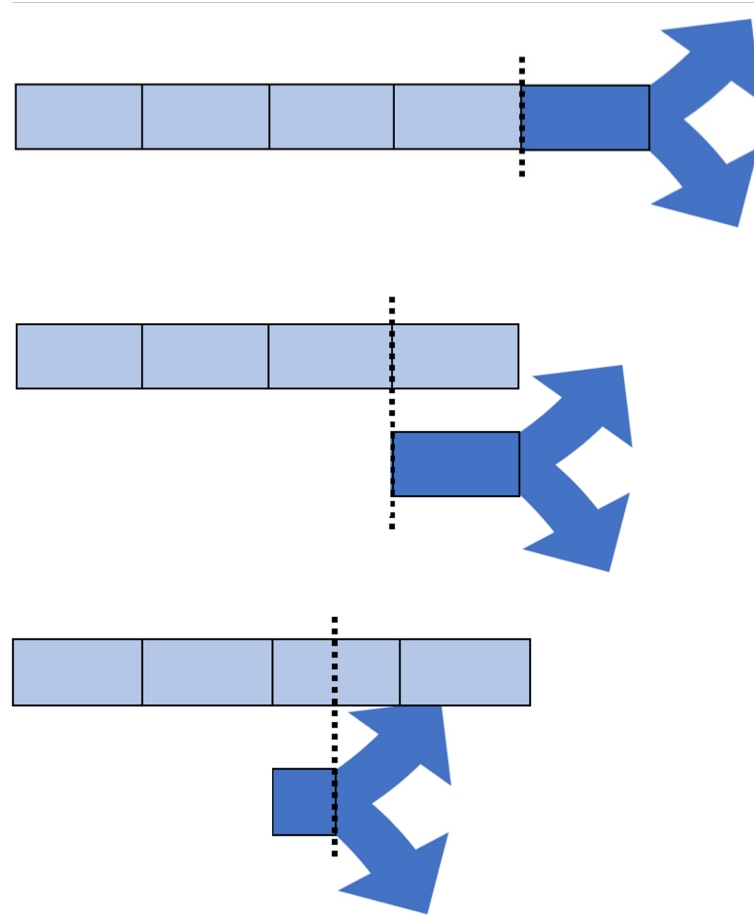
Next-Generation Quantitative Reasoning, Algebra, and Statistics (QAS) assesses the following knowledge and skills:

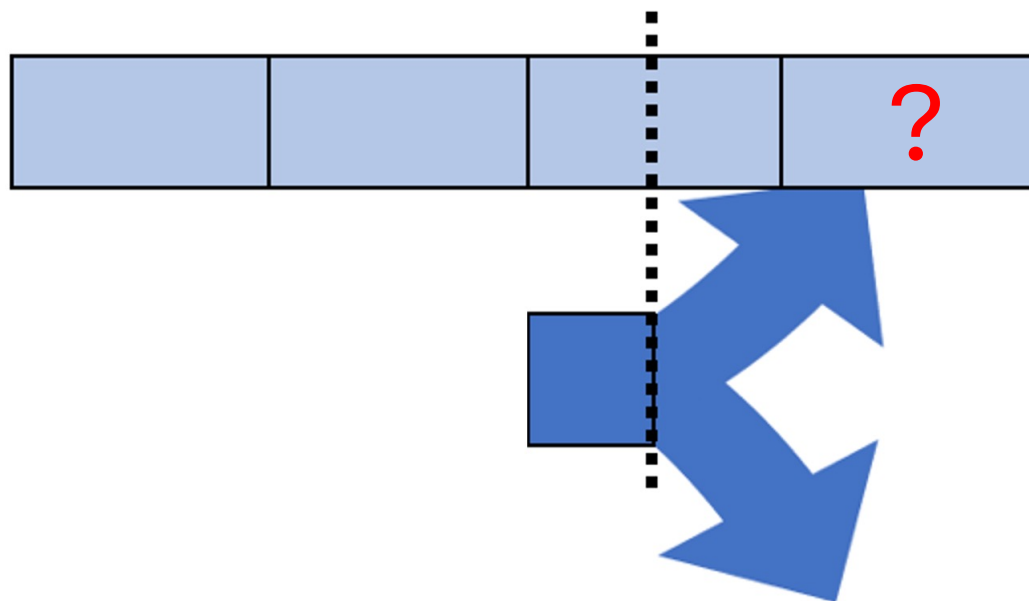
- Rational numbers
- Ratio and Proportional Relationships
- Exponents
- Algebraic Expressions
- Linear Equations
- Linear Applications and Graphs
- Probability Sets
- Descriptive Statistics
- Geometry Concepts

“...the mathematics that most enables students to be successful in college courses is not high school mathematics, but middle school mathematics, especially arithmetic, ratio, proportion, expressions and simple equations.”



An oversimplified, yet illustrative picture:





Algebra II

What is it and what do we do with it?

What's a high school supposed to do?

What will it mean to be “college ready” in mathematics in 2030?

“OK, we get it. We are talking past each other and making a lot of assumptions about shared meanings. But what does that have to do with mathematical meanings and meaningful mathematics?”

Mathematical Meanings?

Fractions

What is a fraction?

Is a fraction one number or two numbers?

Angles

What is an angle?

What attribute are we measuring when we measure angles?

Division

- What different things does division do?
- How can you make sense of $12 \div 3$?
- How can you make sense of $12 \div \frac{1}{3}$?
- How can you make sense of $12 \div -\frac{1}{3}$?

You have three broomsticks

The RED broomstick is three feet long

The YELLOW broomstick is four feet long

The GREEN broomstick is six feet long



$$3 \neq 2$$

The RED broomstick is three feet long



The YELLOW broomstick is four feet long



The GREEN broomstick is six feet long



How much longer is the GREEN broomstick than the RED broomstick?

3 FEET

2 TIMES AS
LONG

The RED broomstick is three feet long



The YELLOW broomstick is four feet long



The GREEN broomstick is six feet long



1. How much longer is the GREEN broomstick than the RED broomstick?

3 FEET	2 TIMES AS LONG
--------	-----------------

2. How much longer is the YELLOW broomstick than the RED broomstick?

1 FOOT	?
--------	---

3. The GREEN broomstick is $\frac{6}{4}$ times as long as the YELLOW broomstick.



6 copies of $\frac{1}{4}$ of yellow
 $6 \cdot \frac{1}{4} = \frac{6}{4}$

The YELLOW broomstick is $\frac{4}{6}$ times as long as the GREEN broomstick.



4 copies of $\frac{1}{6}$ of green
 $4 \cdot \frac{1}{6} = \frac{4}{6}$

The YELLOW broomstick is $\frac{4}{3}$ times as long as the RED broomstick.



$4 \cdot \frac{1}{3} = \frac{4}{3}$

The RED broomstick is $\frac{3}{4}$ times as long as the YELLOW broomstick.



$3 \cdot \frac{1}{4} = \frac{3}{4}$

A certain stock started at a value \$74. One year later it was valued at \$89.54. By what percent did the stock's value increase?

$$\frac{89.54}{74}$$

How many COPIES
OF 74 ARE IN
89.54? 1.21

Measurement

What do we mean when we talk about “measurement”?

Is area a measure?

...or is it something we can measure?

True or False:

- The only way to measure area is with another area

Measurement

“Technically, a *measurement* is a number that indicates a comparison between the attribute of an object being measured and the same attribute of a given unit of measure.”

–Van de Walle (2001)

But what is meant by “comparison”?

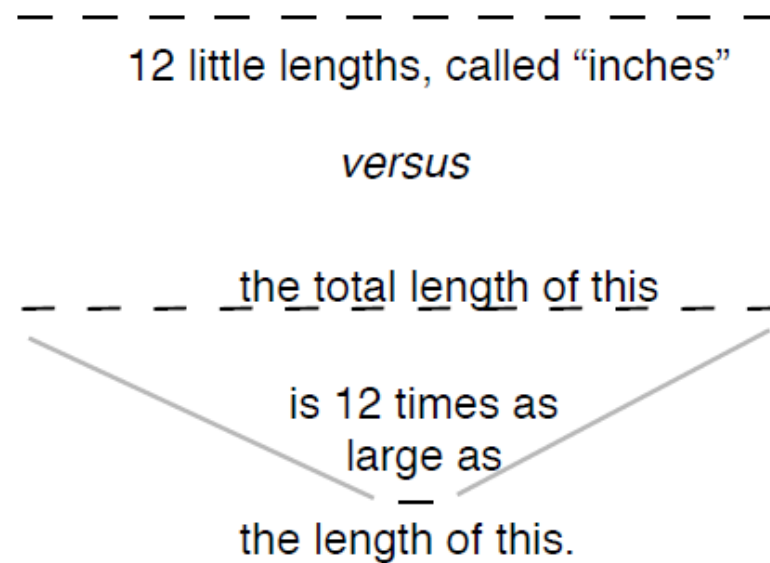


Figure 4. Measure as a number of things vs. measure as a ratio comparison.

Measurement

How about this?

- Determine the attribute you want to measure
- Find something else with the same attribute. Use it as the measuring unit.
- Compare the two: *multiplicatively*.

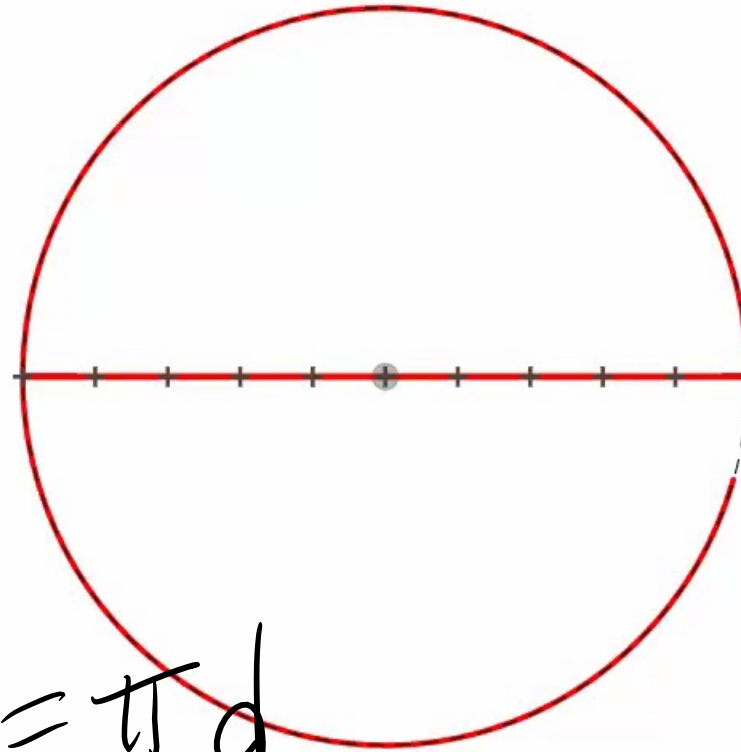
Adjust Radius —●—

Pull Diameters ●—

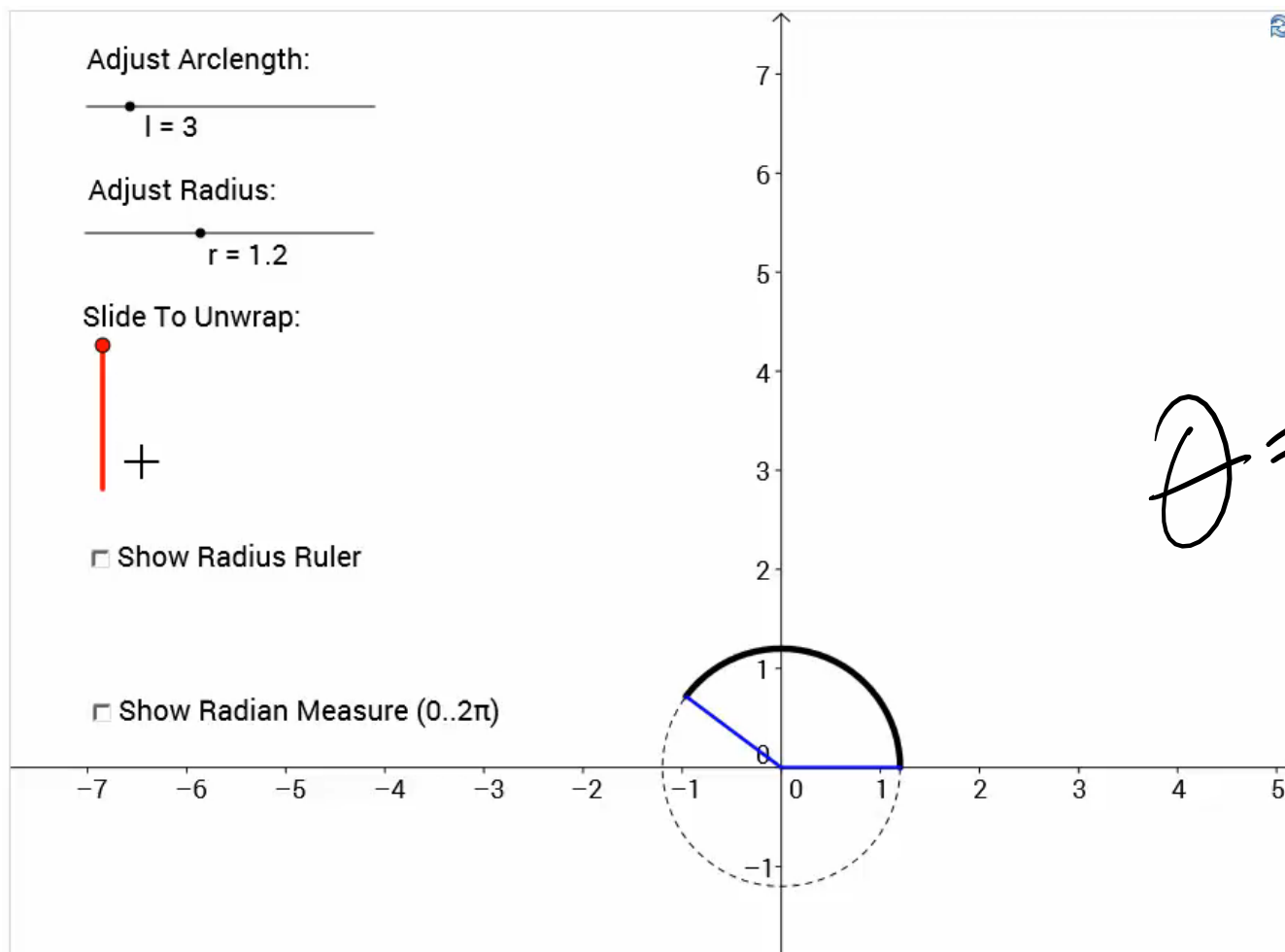
Spin the bit ●—

$$\frac{C}{d} = \pi$$

$$C = \pi d$$



- The circumference is three and a bit times as large as the diameter.



Rise over run

- Why divide?
- Related: What's the difference between a rate and a ratio?

Proportional Relationships

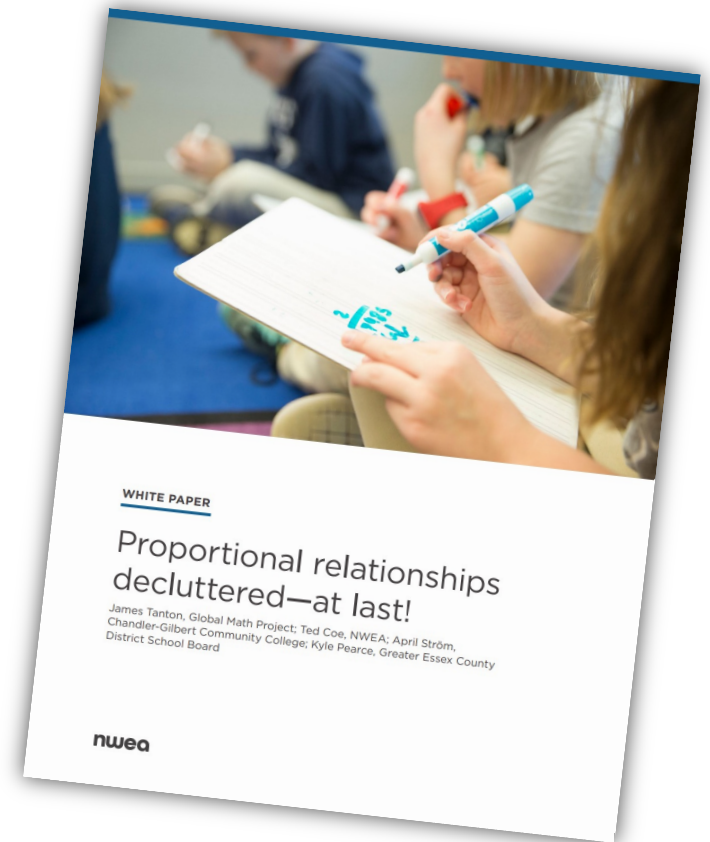
- What is a proportional relationship?
- What does it mean in general to say that one quantity is proportional to another quantity?*

“The confusing jumble of responses here is disturbing. At the very least it points to a lack of a common understanding within the school mathematics community of this very basic and important subject.”

Proportional Relationships Decluttered

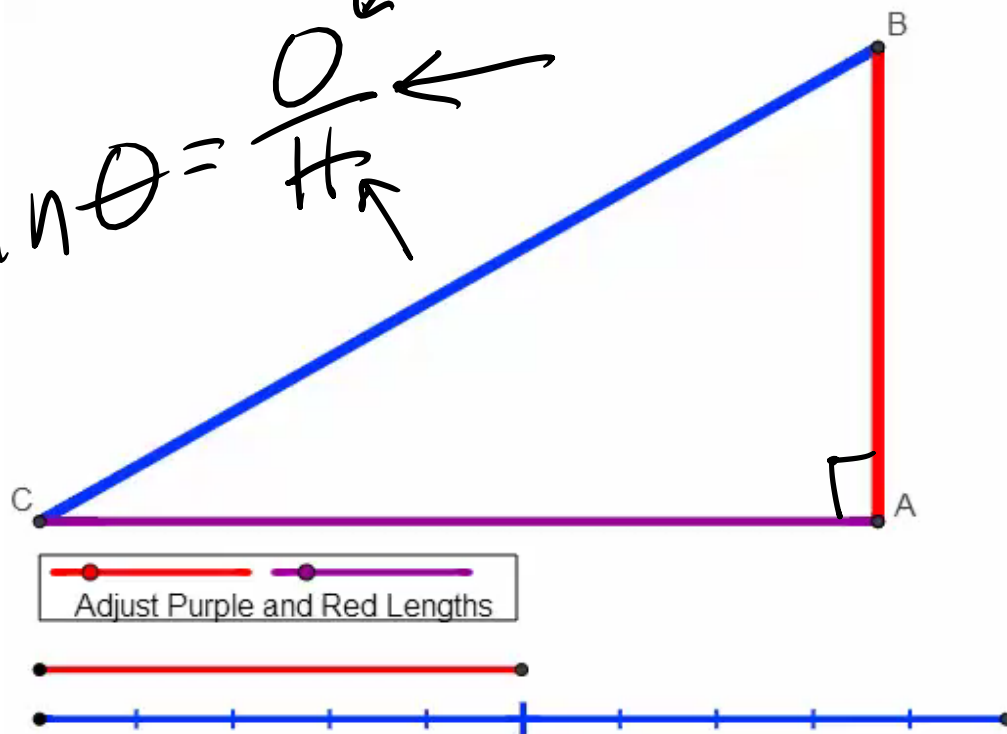
Many common scenarios in the world, in everyday life, and in mathematics involve two or more quantities that we can naturally measure,

- whose measures can or do adopt a variety of possible values,
- whose measures seem related to each other, and
- the measures “scale in tandem.”



$$\sin \theta = \frac{O}{H}$$

Handwritten labels: 'O' is labeled with an arrow pointing to the opposite side (red), and 'H' is labeled with an arrow pointing to the hypotenuse (blue).

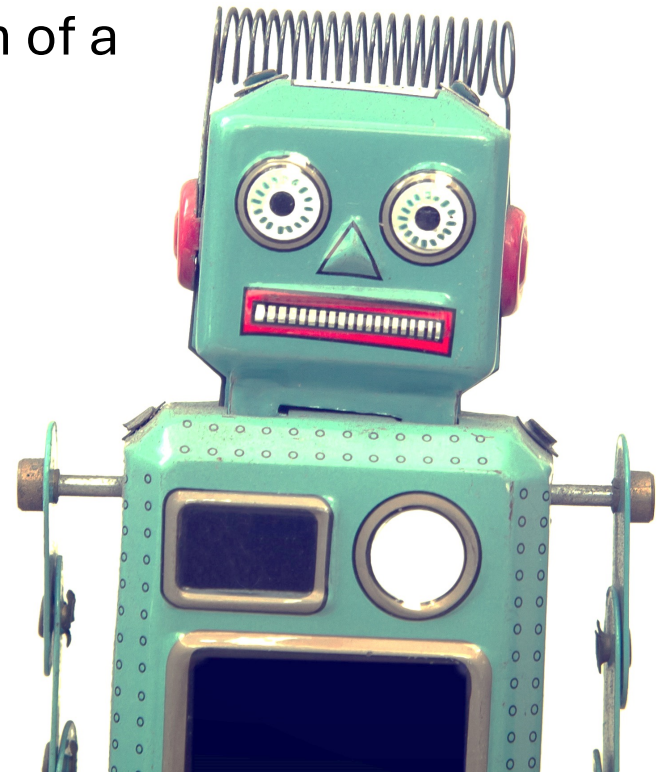


Suppose you have a bank account that grows from \$60 to \$103.68 over three years.

- If the account is growing linearly, what can you tell me? For now, try to stay away from standard formulas. Stick to the quantities themselves.
- If the account is growing exponentially, what can you tell me? As before, stay away from any formulas and work with the quantities themselves.

	Linear (additive)
Total Change:	

- What is a function?
- What is a graph of a function?
- Why would I ever want to “transform” a graph of a function?
- How are function transformations normally taught? (Meanings?)
- And don't get me started on inverse function



Pointless Points and Plotless Plots

“I wanna rock and roll all night, and part of every day .”

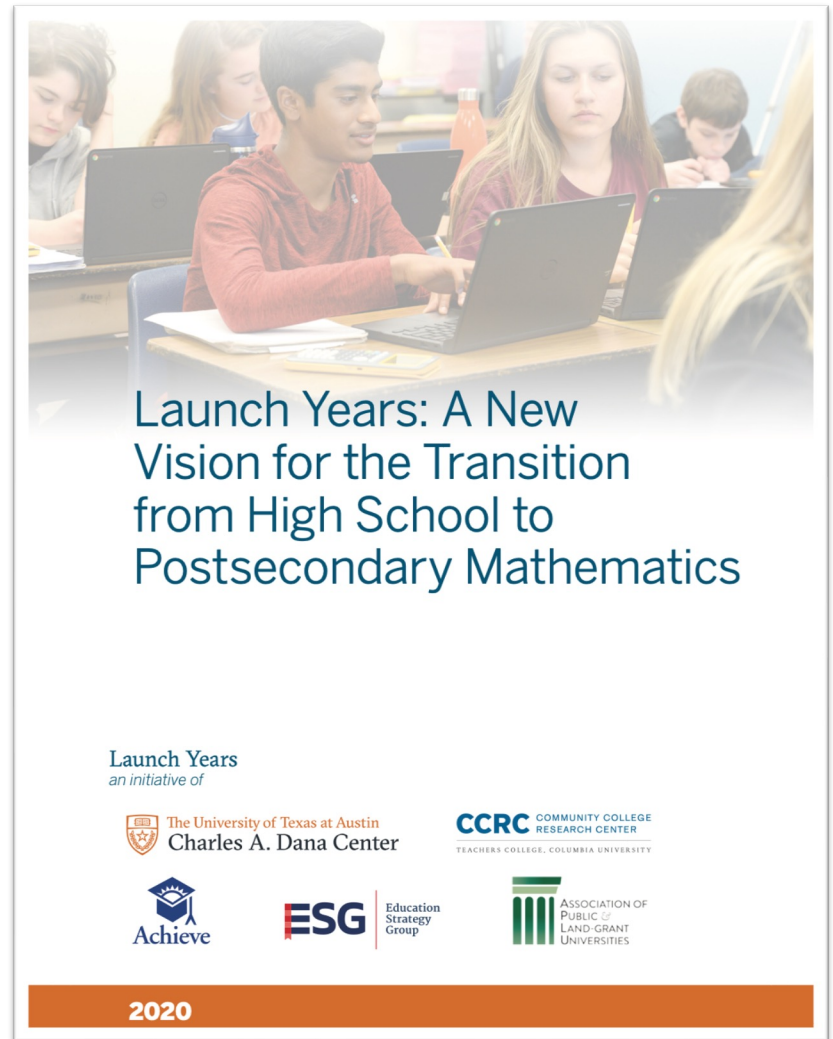
-not KISS

“OK, fine. We have a bit of a mess on our hands when it comes to shared mathematical meanings.

But what is meaningful mathematics?

At a high level...

- Pathways: Finding the right math for students.



NCTM Teaching Practices

Effective Mathematics Teaching Practices

Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.

Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.

Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Meaningful mathematics
discourse must center
on mathematical
meanings!



Photo credit: <https://www.pexels.com/photo/assorted-puzzle-game-1586951/>

A review

- What does rigor mean? What does a rigorous experience require?
 - What does it mean to be mathematically proficient?
 - What should it mean to be college ready?
-

- Mathematical meanings? Check the basics:
 - Fractions, angles, division, measurement, proportional relationships... “it”
- What is meaningful mathematics?

Do your students think:

- Math is a collection of disconnected facts and algorithms
 - **Math is coherent and founded on relationships of ideas**
- Math is to be memorized as given
 - **Math is about constructing knowledge**
- Math is idiosyncratic and situation specific
 - **Math is logical and systematic**
- Math is about quick answer-getting
 - **Math is about authentic problem solving**
- My effort has no impact on my ability to learn math: I am not a math person.
 - **My achievement depends on my persistence**

Thank you!

- ted@coequalmath.com
- tedcoe.com (classroom ready stuff)
- coequalmath.com